LATTICE EMBEDDINGS IN THE RECURSIVELY ENUMERABLE TRUTH TABLE DEGREES

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ABSTRACT. It is shown that every finite lattice, and in fact every recursively presentable lattice, can be embedded in the r.e. tt-degrees by a map preserving least and greatest elements. The decidability of the 1-quantifier theory of the r.e. tt-degrees in the language with $\leqslant$, $\vee$, $\wedge$, 0, and 1 is obtained as a corollary.

Introduction. A set $A \subseteq \omega$ is truth table (tt) reducible to $B \subseteq \omega$ ($A \leqslant_{tt} B$) if answers to questions of the form "$n \in A$?" are given by a finite Boolean combination, effectively determined from n, of answers to questions of the form "$k \in B$?". Sets A and B are of the same tt-degree if $A \leqslant_{tt} B$ and $B \leqslant_{tt} A$. We consider the structure consisting of the tt-degrees of recursively enumerable sets of natural numbers. Odifreddi [4] and Rogers [6] contain background information on the tt-degrees. In particular, Fejer and Shore [1] contains information about the r.e. tt-degrees and about questions relating to the decidability of the theory of the r.e. tt-degrees. They show there that every recursively presentable lattice can be embedded in the r.e. tt-degrees preserving least element. Using this, they show that the ∃ theory of the r.e. tt-degrees in the language with $\leqslant$, $\vee$, $\wedge$, 0 is decidable, and ask whether the ∃ theory is still decidable when 1 is added to the language. This decidability question can be answered by determining which finite lattices can be embedded in the r.e. tt-degrees preserving least and greatest elements. Jockusch and Mohrherr [3] have shown that the diamond lattice, the pentagon lattice, and the 1-n-1 lattices can be embedded preserving least and greatest elements, but leave open the general question, and even such special cases as the three generator Boolean algebra. The embedding used in their proof requires that the lattice in question have the property that no element which is the inf of a pair of incomparable elements of the lattice can be joined up to the 1 of the lattice (except by 1 itself). We show here that all finite lattices, and in fact all recursively presentable lattices, can be embedded in the r.e. tt-degrees preserving least and greatest elements (provided the lattice has distinct least and greatest elements). Our proof for the general lattices combines a generalization of the coding method used by Jockusch and Mohrherr with the strategy for preserving nonzero infs used by Fejer and Shore.

We prove first, in §1, that every lattice with a finite representation (and distinct least and greatest elements) can be embedded, and then in §2 outline the modifications needed to embed a recursively presented (possibly infinite) lattice. As a

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corollary to this, we then show that the existential theory of the r.e. tt-degrees in the language with $\leqslant$, $\vee$, $\wedge$, 0, 1 is decidable.

Our notation is standard and follows that of Fejer and Shore [1] very closely. We make implicit use of the convention that if $\{e\}_s(x)\downarrow$, then $x < s$. The material in this paper appears as part of my doctoral dissertation [2]. I am grateful to Richard Shore for his guidance.

1. Finite lattices. We first explain our notation and outline some of the ideas which will be used throughout the proof. The organization of the proof follows that of Fejer and Shore [1].

If $\{e\}(x)\downarrow$, then $[e](x)$ is the truth table with index $\{e\}(x)$, and $|[e](x)|$ is the size of the truth table. If A is a set of natural numbers, then we define $[e]^A$ by

$$[e]^A(x) = \begin{cases} 1 & \text{if } \{e\}(x)\downarrow \text{ and } A \text{ satisfies } [e](x), \\ 0 & \text{if } \{e\}(x)\downarrow \text{ and } A \text{ does not satisfy } [e](x), \\ \uparrow & \text{if } \{e\}(x)\uparrow, \end{cases}$$

and $[e]_s^A$, by

$$[e]_s^A(x) = \begin{cases} [e]^A(x) & \text{if } \{e\}_s(x)\downarrow, \\ \uparrow & \text{if } \{e\}_s(x)\uparrow. \end{cases}$$

Sometimes we write $[e]^{A_s}$ for $[e]_s^{A_s}$, to simplify notation. So $A \leqslant_{\text{tt}} B$ iff for some e, $A = [e]^B$.

Let $\mathscr{P} = \langle P, \leqslant, \vee, \wedge, 0_{\mathscr{P}}, 1_{\mathscr{P}} \rangle$ be a finite lattice with $P = \{ p_i : i \in [0, n] \}$ and $p_0 = 0_{\mathscr{P}}$, $p_n = 1_{\mathscr{P}}$. A representation of $\mathscr{P}$ will be a set $\Phi = \{ \alpha_r : r \in J \}$, where for $r \in J$, $\alpha_r : [0, n] \to \omega$, and for all i, j, $k \leqslant n$ the following hold:

(1) $p_i \leqslant p_j \leftrightarrow (\forall r_1, r_2 \in J)[\alpha_{r_1}(j) = \alpha_{r_2}(j) \to \alpha_{r_1}(i) = \alpha_{r_2}(i)]$,

(2) $p_i \vee p_j = p_k \leftrightarrow (\forall r_1, r_2 \in J)$

$$[\alpha_{r_1}(i) = \alpha_{r_2}(i) \text{ and } \alpha_{r_1}(j) = \alpha_{r_2}(j) \to \alpha_{r_1}(k) = \alpha_{r_2}(k)],$$

(3) $p_i \wedge p_j = p_k \leftrightarrow (\forall r_1, r_2 \in J)[\alpha_{r_1}(k) = \alpha_{r_2}(k) \to \exists m \in \omega$

$$\exists s_0, \ldots, s_m \in J(s_0 = r_1 \text{ and } s_m = r_2 \text{ and for } h < m - 1, h \text{ even,}$$

$$\alpha_{s_h}(i) = \alpha_{s_{h+1}}(i) \text{ and } \alpha_{s_{h+1}}(j) = \alpha_{s_{h+2}}(j))].$$

For example, consider the lattice

$$\mathscr{P} = (\{ p_0, p_1, p_2, p_3, p_4, p_5 \}, \leqslant, \vee, \wedge, 0_{\mathscr{P}}, 1_{\mathscr{P}})$$

with $0_{\mathscr{P}} = p_0$, $1_{\mathscr{P}} = p_5$ and the ordering, join and meet given by the following diagram. Notice that $\mathscr{P}$ is a simple example of a lattice which cannot be embedded using Jockusch and Mohrherr's proof [3]. An example of a representation for $\mathscr{P}$ is the set of rows $\Phi = \{ \alpha_r : r \in [0, 6] \}$, where $\alpha_r : [0, 5] \to [0, 6]$, and the values for $\alpha_r(i)$ for $r \in [0, 6]$ and $i \in [0, 5]$ are given by Table 1.
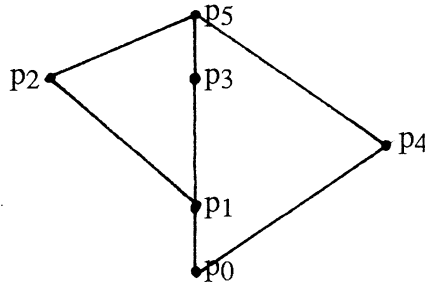
Rows 0, 1, 2, and 3 establish (1) for all of the relationships $p_i \not\leqslant p_j$ which hold in $\mathscr{P}$, and rows 4, 5, and 6 give the interpolants whose existence is asserted by (3). The table maintains (1) and (2) for all of the relationships $p_i \leqslant p_j$ and $p_i \vee p_j = p_k$ which hold in $\mathscr{P}$.

TABLE 1

		i					
		0	1	2	3	4	5
	0	0	0	0	0	0	0
	1	0	1	1	1	0	1
	2	0	1	2	1	2	2
r	3	0	1	2	3	3	3
	4	0	4	4	4	3	4
	5	0	5	5	4	0	5
	6	0	0	0	6	2	6

A representation is called finite if J (the set of indices for the rows) is finite. Pudlak and Tuma [5] have shown that every finite lattice has a finite representation. In this section we consider only finite lattices, so we use their result and assume that we have a finite representation for the lattice we embed. This will simplify the construction of the embedding somewhat. In §2 we will prove that every lattice with a recursive presentation can be embedded preserving least and greatest elements. It is easy to see that every finite lattice has a recursive presentation. It will then not be necessary to use the difficult result of Pudlak and Tuma. So we assume, without loss of generality, that for some $d \in \omega$, $J = [0, d]$, $n > 0$, and for r_1, $r_2 \leqslant d$, $\alpha_{r_1}: [0, n] \to [0, d]$, $\alpha_{r_1}(0) = 0$, and $(r_1 \neq r_2 \to \alpha_{r_1}(n) \neq \alpha_{r_2}(n))$.

As in Fejer and Shore [1], our construction will define a recursive function, $f(x, s)$, where $f: \omega \times \omega \to [0, d]$. We define r.e. sets, A_i, $i \leqslant n$ by

$$A_{i,s} = \left\{ \langle x, \alpha_{f(x,s')}(i) \rangle : x \in \omega \text{ and } s' \leqslant s \right\}$$

and

$$A_i = \left\{ \langle x, \alpha_{f(x,s)}(i) \rangle : x \in \omega \text{ and } s \in \omega \right\}.$$

The map $p_i \to \deg_{tt}(A_i)$ will provide the embedding of $\mathscr{P}$ into the r.e. tt-degrees.

Our construction will ensure the following:

(4) A_0 is recursive.

(5) $p_i \leqslant p_j \to A_i \leqslant_{tt} A_j.$

(6) $p_i \vee p_j = p_k \to A_k \leqslant_{tt} A_i \oplus A_j.$

(7) $K \leqslant_{tt} A_n$, where K is a fixed complete r.e. set.

(8) $p_i \not\leqslant p_j \to A_i \not\leqslant_{tt} A_j.$

(9) $p_i \not\leqslant p_j$ and $p_j \not\leqslant p_i$ and $p_i \wedge p_j = p_k \to$ if ($Y \leqslant_{tt} A_i$ and $Y \leqslant_{tt} A_j$) then $Y \leqslant_{tt} A_k.$

Our definition of A_0 will guarantee that (4) holds, since for all $r \leqslant d$, $\alpha_r(0) = 0$, so $A_0 = \{\langle x, \alpha_r(0)\rangle : x \in \omega, r \in J\} = \{\langle x, 0\rangle : x \in \omega\}$ is recursive. The definition of the A_i will also guarantee that (5) holds. Suppose $p_i \leqslant p_j$. Then

$$\langle x, y\rangle \in A_i \leftrightarrow \exists s\big(\alpha_{f(x,s)}(i) = y\big)$$

$$\leftrightarrow \exists r \leqslant d\big(\langle x, \alpha_r(j)\rangle \in A_j \text{ and } \alpha_r(i) = y\big).$$

Condition (6) will be established by a simple restriction on the function f. We insist that

(10)   For each x there is at most one s such that $f(x, s + 1) \neq f(x, s)$.

Then if $p_i \vee p_j = p_k$, we get $A_k \leqslant_{tt} A_i \oplus A_j$ as follows:

(11) $\langle x, y\rangle \in A_k \leftrightarrow y = \alpha_{f(x,0)}(k)$

or $\big(y \neq \alpha_{f(x,0)}(k)$ and $\exists s > 0 \; y = \alpha_{f(x,s)}(k)\big).$

Condition (2) allows us to recognize the right side of (11) in a tt-way using $A_i \oplus A_j$. If $\alpha_{f(x,0)}(k) \neq \alpha_{f(x,s)}(k)$, then by (2), either $\alpha_{f(x,0)}(i) \neq \alpha_{f(x,s)}(i)$ or $\alpha_{f(x,0)}(j) \neq \alpha_{f(x,s)}(j)$. Hence from (11) we get

(12) $\langle x, y\rangle \in A_k \leftrightarrow y = \alpha_{f(x,0)}(k)$

or $\exists r \leqslant d\big[y = \alpha_r(k)$ and $\langle x, \alpha_r(i)\rangle \in A_i$

and $\langle x, \alpha_r(j)\rangle \in A_j$ and $\forall z \leqslant d$

$\big((\langle x, z\rangle \in A_i \to z = \alpha_r(i)$ or $z = \alpha_{f(x,0)}(i))$

and $(\langle x, z\rangle \in A_j \to z = \alpha_r(j)$ or $z = \alpha_{f(x,0)}(j))\big)\big]$

and this provides the desired tt-reduction from $A_i \oplus A_j$.

The construction will guarantee that (7), (8) and (9) hold. Before describing this in detail, we need more notation. Let $h \in \omega$ be such that there are h many distinct pairs, (p_i, p_j) in $\mathscr{P}$ such that $p_i \not\leqslant p_j$. Let $\{"p_{i_a} \not\leqslant p_{j_a}" : 1 \leqslant a \leqslant h\}$ be a listing of all such relations. To establish (8) we will satisfy the requirements.

(13) $P_e^a : A_{i_a} \neq [e]^{A_{j_a}}$,   for $1 \leqslant a \leqslant h$ and $e \in \omega$.

Let $m \in \omega$ be such that there are $(m - h)$ many distinct triples, (p_i, p_j, p_k), in $\mathscr{P}$ such that $p_i \not\leqslant p_j$ and $p_j \not\leqslant p_i$ and $p_i \wedge p_j = p_k$. Let $\{$"$p_{i_a} \wedge p_{j_a} = p_{k_a}$": $h + 1 \leqslant a \leqslant m\}$ be a listing of all such relations. To establish (9) we will satisfy the requirements

$$(14) \quad Q_e^a: [e]^{A_{i_a}} = [e]^{A_{j_a}} = Y \to Y \leqslant_{\mathrm{tt}} A_{k_a} \quad \text{for } h + 1 \leqslant a \leqslant m \text{ and } e \in \omega.$$

Next we recursively partition ω and associate elements of the partition so that

(15) Each P_e^a has an infinite set of numbers associated with it. Let $D(\langle e, a \rangle)$ be the set of numbers associated with P_e^a.

(16) Each $x \in \omega$ has two finite sets of numbers associated with it, call them $W(x)$ and $CW(x)$. Each element of $W(x) \cup CW(x)$ is greater than x, and $\min(CW(x)) > \max(W(x))$ and

$$|W(x)| = 2^{x+1}(h + 2(m - h)) \quad \text{and}$$
$$|CW(x)| = 2(m - h)(2^{x+1})(h + 2(m - h)).$$

For each $x \in \omega$ we further partition $W(x)$ into $h + 2(m - h)$ many pieces, each of size 2^{x+1}, and label these so that

$$W(x) = \bigcup_{a \in [1,h]} W^a(x) \cup \left(\bigcup_{(a,k) \in [h+1,m] \times \{1,2\}} W^{(a,k)}(x) \right),$$

where for $a \in [1, h]$, $|W^a(x)| = 2^{x+1}$ and for $(a, k) \in [h + 1, m] \times \{1, 2\}$, $|W^{(a,k)}(x)| = 2^{x+1}$. If $a \in [1, h]$ and $y \in W^a(x)$, we sometimes say that y is an element of $W(x)$ "reserved for a", similarly for $(a, k) \in [h + 1, m] \times \{1, 2\}$.

We use the elements of $W(x)$ to witness $x \in K$. When we act to code x, our action will be chosen to cooperate with some requirement P_e^a or Q_e^a. One witness will be used when we cooperate with P_e^a; two witnesses may be needed when we cooperate with Q_e^a. Since there are h many different kinds of P_e^a requirements, and $m - h$ many different kinds of Q_e^a requirements, we have $h + 2(m - h)$ many different sets of witnesses for coding x. Other requirements in the construction will be allowed to use up elements of our witness sets; we will use an inductive argument to show that 2^{x+1} elements for each requirement will be enough. The elements of $CW(x)$ serve as partners to the elements of $W(x)$, and are used to witness $x \notin K$. Each element of $W(x)$ will need two partners for each kind of inf requirement, hence we make $|CW(x)| = 2(m - h)|W(x)|$. For each $x \in \omega$ and each triple $(y, a, k) \in W(x) \times [h + 1, m] \times \{1, 2\}$ we associate a unique element of $CW(x)$. We express this association via a recursive 1-1 function g such that for each $x \in \omega$,

$$g: W(x) \times [h + 1, m] \times \{1, 2\} \overset{\text{onto}}{\to} CW(x)$$

and we call $g(y, a, k)$ a partner of y.

We now describe the strategies for satisfying each of (7), P_e^a and Q_e^a in isolation; after that we describe how these requirements interact and cooperate with each other.

To get $K \leqslant_{tt} A_n$, we establish

(17) $x \in K \leftrightarrow \exists y \in W(x)\big[\exists r \leqslant d\,(r \neq f(y,0) \text{ and } \langle y, \alpha_r(n)\rangle \in A_n)$

$$\text{and } \forall (q,a,k) \in [0,d] \times [h+1,m] \times \{1,2\}$$

$$\big(\langle g(y,a,k), \alpha_q(n)\rangle \in A_n \rightarrow q = f(g(y,a,k),0)\big)\big].$$

Condition (17) says that x is in K if and only if there is a $y \in W(x)$ such that at some stage s, $f(y,s) \neq f(y,0)$, but for all z such that z is a partner of y, and all stages s', $f(z,s') = f(z,0)$. Let $\{k_t: t \in \omega\}$ be a 1-1 recursive enumeration of K, and $K_s = \{k_t: t < s\}$. During odd stages of the construction we will act for the sake of coding; during even stages of the construction we will act for the sake of other requirements. At stage $s + 1 = 2t + 1$ we will try to establish (17) for $x = k_t$. We choose a $y \in W(k_t)$ and an $r \leqslant d$ such that $f(y,2t) = f(y,0)$, $r \neq f(y,0)$, and for all $(a,k) \in [h+1,m] \times \{1,2\}$, $f(g(y,a,k),2t) = f(g(y,a,k),0)$. The exact choice of y and r will depend on the interaction between coding and other requirements. We then set $f(y,2t+1) = r$ (thereby enumerating $\langle y, \alpha_r(n)\rangle$ into A_n) and insist that

(18) $\forall s' \geqslant 2t + 1 \,\forall z \in W(k_t) \cup CW(k_t)(f(z,s') = f(z,,2t+1)).$

In this way, " $\rightarrow$ " of (17) is established.

At stage $s' + 1 = 2t'$ of the construction, it may be the case for some x that $x \notin K_{s'+1}$, but $\exists y \in W(x)(f(y,s'+1) \neq f(y,s'))$. To maintain (17), for any such y we insist that

$$\exists (r,a,k) \in [0,d] \times [h+1,m] \times \{1,2\}$$

$$(f(g(y,a,k),s'+1) = r \text{ and } r \neq f(g(y,a,k),0)).$$

The schemes for cooperation among the requirements will guarantee that each of these actions is possible.

We will use numbers in the set $D(\langle e,a\rangle)$ to satisfy the requirements P_e^a, for $a \in [1,h]$, $e \in \omega$. Since $p_{i_a} \not\leqslant p_{j_a}$, by (1) we get that there are $r(a)_1$, $r(a)_2 \leqslant d$ such that for $a \in [1,h]$

(19) $\alpha_{r(a)_1}(j_a) = \alpha_{r(a)_2}(j_a)$ and $\alpha_{r(a)_1}(i_a) \neq \alpha_{r(a)_2}(i_a).$

For each $y \in D(\langle e,a\rangle)$ we set $f(y,0) = r(a)_1$. At stage $s + 1 = 2t$ during the construction, if there is an $x \in D(\langle e,a\rangle)$ such that $f(x,s) = f(x,0) = r(a)_1$ and

$$[e]^{A_{j_a,s}}\big(\langle x, \alpha_{r(a)_2}(i_a)\rangle\big)\downarrow = 0,$$

then P_e^a tries to set $f(x,s+1) = r(a)_2$, and to preserve $A_{j_a,s} \restriction \|[e](\langle x, \alpha_{r(a)_2}(i_a)\rangle)\|$. In this way we enumerate $\langle x, \alpha_{r(a)_2}(i_a)\rangle$ into $A_{i_a,s+1}$ and cause

$$A_{i_a,s+1}\big(\langle x, \alpha_{r(a)_2}(i_a)\rangle\big) \neq [e]^{A_{j_a,s+1}}\big(\langle x, \alpha_{r(a)_2}(i_a)\rangle\big).$$

Next, we consider a requirement Q_e^a, for $a \in [h + 1, m]$ and $e \in \omega$. We try to make $[e]^{A_{i_a}} \neq [e]^{A_{j_a}}$; if this fails then we must be able to recover $Y = [e]^{A_{i_a}} = [e]^{A_{j_a}}$ from A_{k_a} in a tt way. Suppose s and y are such that $\{e\}_s(y')\downarrow$ for all $y' \leq y$. At stage s there are two cases to consider. First, $[e]^{A_{i_a,s}}(y') \neq [e]^{A_{j_a,s}}(y')$ for some $y' \leq y$, w.l.o.g. $y' = y$. In this case we try to maintain the inequality on y by either restraining all numbers $\leq \|[e](y)\|$ from $A_{i_a,s+1}$ and $A_{j_a,s+1}$ if possible, or by causing numbers to be enumerated into $A_{i_a,s+1}$ and $A_{j_a,s+1}$ or both in such a way that the inequality on y remains. This method of enumerating is based on Jockusch and Mohrherr's method of preserving an inf in their embedding of the diamond lattice in the r.e. tt-degrees in [7] and will be described in more detail when we discuss the cooperation between coding and Q_e^a. The second case is $[e]^{A_{i_a,s}}(y') = [e]^{A_{j_a,s}}(y')$ for all $y' \leq y$. We would like to be able to either cause $[e]^{A_{i_a,s+1}}(y) \neq [e]^{A_{j_a,s+1}}(y)$, or to be able to assume that we can recover $[e]^{A_{j_a}}(y)$ from $A_{i_a,s}$, $A_{j_a,s}$ and A_{k_a}. To this end, we consider all numbers which may later be enumerated into A_{i_a} or A_{j_a} affecting the computations $[e]^{A_{i_a,s}}(y)$ and $[e]^{A_{j_a,s}}(y)$. If there is a way of enumerating some subset of these numbers into $A_{i_a,s+1} \cup A_{j_a,s+1}$ so that $[e]^{A_{i_a,s+1}}(y) \neq [e]^{A_{j_a,s+1}}(y)$, then we take this action. If this is impossible, then we will be able to argue, roughly, that all changes in A_{i_a} after stage s which affect the computation $[e]^{A_{i_a}}(y)$ occur because we defined f in a way that also caused changes in A_{j_a}. The lattice representation will then cause all such changes to be reflected in A_{k_a}, and so we will be able to recover $[e]^{A_{i_a}}(y)$ from A_{k_a}.

Let $\{T_i : i \in \omega\}$ be an effective listing of the requirements $\{P_e^a : a \in [1, h], e \in \omega\}$ and $\{Q_e^a : a \in [h + 1, m], e \in \omega\}$. We now describe the schemes for cooperation among the T_i and the coding.

Coding and $T_i = P_e^a$. Since $D(\langle e, a \rangle)$ is disjoint from $\bigcup_{x \in \omega}(W(x) \cup CW(x))$, action taken for the sake of T_i has no bearing on (17), and so has no interaction with the coding. Action taken for coding may interact with P_e^a, though. Suppose that P_e^a has acted before stage $s + 1 = 2t + 1$, and wants to preserve $A_{j_a} \upharpoonright u$. If $k_t \geq u$, then since all elements of $W(k_t)$ are greater than k_t, the action taken for coding will not disturb P_e^a. However, if $k_t < u$, then we would like to choose the coding action in such a way as to preserve $A_{j_a,s} \upharpoonright u$. Let $r(a)_1$, $r(a)_2 \leq d$ be as defined in (19). We claim we can use $r(a)_1$ and $r(a)_2$ to enumerate a number into $A_{n,s+1}$ without disturbing $A_{j_a,s+1}$. For each $x \in \omega$, $a \in [1, h]$ and each $y \in W^a(x)$, we have set $f(y, 0) = r(a)_1$. Now if we decide that coding should cooperate with P_e^a at $s + 1 = 2t + 1$, then we choose a $y \in W^a(k_t)$ such that $f(y, s) = f(y, 0)$ and for all $(a', k) \in [h + 1, m] \times \{1, 2\}$, $f(g(y, a', k), s) = f(g(y, a', k), 0)$, and we set $f(y, s + 1) = r(a)_2$. In this way we enumerate $\langle y, \alpha_{r(a)_2}(n) \rangle$ into $A_{n,s+1}$ and leave $A_{j_a,s+1} = A_{j_a,s}$. It will be necessary for coding to cooperate with P_e^a at most finitely often, since there can only be finitely many t with $k_t < u$.

Coding and $T_i = Q_e^a$. Action taken for coding may interact with Q_e^a. Suppose s and y are such that $\forall y' \leq y (\{e\}(y')\downarrow)$ and $[e]^{A_{i_a,s}} \upharpoonright y = [e]^{A_{j_a,s}} \upharpoonright y$ and $[e]^{A_{i_a,s}}(y) \neq [e]^{A_{j_a,s}}(y)$. If $s + 1 = 2t + 1$ and $k_t \geq \|[e](y)\|$ then action taken for coding at $s + 1$ will not disturb Q_e^a. However, if $k_t < \|[e](y)\|$, then we need to maintain the inequality at y. Since for all $a \in [h + 1, m]$, $p_{i_a} \neq 1_{\mathscr{P}}$ and $p_{j_a} \neq 1_{\mathscr{P}}$, (1) implies that

there must be $r(a,1)_1$, $r(a,1)_2$, $r(a,2)_1$, $r(a,2)_2 \in J$ satisfying

$$
\begin{aligned}
\alpha_{r(a,1)_1}(i_a) &= \alpha_{r(a,1)_2}(i_a) \quad \text{and} \\
\alpha_{r(a,1)_1}(n) &\neq \alpha_{r(a,1)_2}(n) \quad \text{and} \\
\alpha_{r(a,2)_1}(j_a) &= \alpha_{r(a,2)_2}(j_a) \quad \text{and} \\
\alpha_{r(a,2)_1}(n) &\neq \alpha_{r(a,2)_2}(n).
\end{aligned}
$$

(20)

We will use $r(a,k)_1$, $r(a,k)_2$, for $k \in \{1,2\}$, to enumerate into $A_{n,s+1}$ while maintaining Q_e^a's inequality on y. For each $x \in \omega$ and $(a,k) \in [h+1,m] \times \{1,2\}$, and each $y \in W^{(a,k)}(x)$, we set $f(y,0) = r(a,k)_1$. Now if we decide that coding should cooperate with Q_e^a at $s+1 = 2t+1$, then we choose $y_1 \in W^{(a,1)}(k_t)$ and $y_2 \in W^{(a,2)}(k_t)$ such that $f(y_1,s) = f(y_1,0)$, and $f(y_2,s) = f(y_2,0)$ and for all $(a',k') \in [h+1,m] \times \{1,2\}$, $f(g(y_1,a',k'),s) = f(g(y_1,a',k'),0)$ and $f(g(y_2,a',k'),s) = f(g(y_2,a',k'),0)$. We have three possible actions available now;

1. We set $f(y_1, s+1) = r(a,1)_2$ and $f(y_2, s+1) = f(y_2,s)$,
2. We set $f(y_1, s+1) = f(y,s)$ and $f(y_2, s+1) = r(a,2)_2$,
3. We set $f(y_1, s+1) = r(a,1)_2$ and $f(y_2, s+1) = r(a,2)_2$.

The proof of Lemma 1 shows that one of these three courses of action must maintain the inequality on y.

Actions taken by Q_e^a also interact with the coding. Q_e^a's action at $s+1$ consists of looking ahead at numbers which may later get enumerated into $A_{i_a} \cup A_{j_a}$ and possibly putting some of these numbers into $A_{i_a,s+1} \cup A_{j_a,s+1}$. For each x, we must make sure that Q_e^a's action does not use up all the elements of $W(x)$ before the stage $s = 2t+1$ such that $x = k_t$ (if such a stage exists). To ensure this we place restrictions on the numbers that Q_e^a is allowed to enumerate. We say that "Q_e^a has access to x" if we are allowed to define $f(y,s+1) \neq f(y,s)$ for $y \in W(x) \cup CW(x)$ for the sake of Q_e^a. If $Q_e^a = T_i$ and $x \leqslant i$, then we do not allow Q_e^a to have access to x. For $b \in [1,h] \cup ([h+1,m] \times \{1,2\})$ we define

$$
\begin{aligned}
G(x,b,i,s+1) = \big\{ y \in W^b(x) &: T_i \text{ caused } f(y,s+1) \neq f(y,s) \text{ or} \\
&\exists (a',k') \in [h+1,m] \times \{1,2\} \\
&(T_i \text{ caused } f(g(y,a',k'),s+1) \neq f(g(y,a',k'),s)) \big\}.
\end{aligned}
$$

Then if $x > i$ and $b \in [1,h] \cup ([h+1,m] \times \{1,2\})$, we restrict T_i's action so that $|\bigcup_{s \in \omega} G(x,b,i,s)| \leqslant 2^{x-i}$. This is done via the following rules. At stage $s+1 = 2t$, if $x \in K_t$, then we allow no requirement Q_e^a access to x, and in this way ensure that $G(x,b,i,s) = \varnothing$ for all b, i. If $x \notin K_t$ and $G(x,b,j,s') = \varnothing$ for all $j \leqslant i$, $s' \leqslant s$, and $b \in [1,h] \cup ([h+1,m] \times \{1,2\})$, then we allow T_i to have access to x (provided $i < x$) and so we allow T_i to change f at $s+1$ on a subset of $W(x) \cup CW(x)$, so long as

$$
|G(x,b,i,s+1)| \leqslant 2^{x-i}, \quad \text{for all } b \in [1,h] \cup ([h+1,m] \times \{1,2\}).
$$

We use the process of "freezing x for T_i" to ensure that $|\bigcup_{s \in \omega} \bigcup_{i \in \omega} G(x,b,i,s)| \leqslant 2^{x+1} - 2$. For i, $s \in \omega$, if

$$
\exists j \leqslant i \, \exists s' \leqslant s \, \exists b \in [1,h] \cup ([h+1,m] \times \{1,2\})(G(x,b,j,s') \neq \varnothing)
$$

then we say that "x is frozen for T_i at t", for all $t \geqslant s$. This means that at all $t \geqslant s$, T_i does not have access to x, and so $G(x, b, i, t)$ will be empty. In this way we will ensure that the maximum number of elements of $W^b(x)$, for any $x \in \omega$, $b \in [1, h]$ $\cup ([h + 1, m] \times \{1, 2\})$, used up by requirements other than coding is

$$\left| \bigcup_{s \in \omega} \bigcup_{i \in \omega} G(x, b, i, s) \right| \leqslant \sum_{i \in \omega} \left(\left| \bigcup_{s \in \omega} G(x, b, i, s) \right| \right)$$

$$\leqslant \sum_{i < x} 2^{x-i} = 2^{x+1} - 2.$$

Since $W^b(x)$ contains 2^{x+1} elements, there will always be enough for coding. The restrictions placed on Q_e^a's action will be expressed more precisely in the definition of a legal move for Q_e^a.

THEOREM 1. *If* $\mathscr{P} = \langle P, \leqslant, \vee, \wedge, 0_{\mathscr{P}}, 1_{\mathscr{P}} \rangle$ *is a lattice with more than one element, and* $\mathscr{P}$ *has a finite representation, then P can be embedded into the r.e. tt-degrees by a map which preserves zero and one.*

PROOF. Let $P = \{ p_0, \ldots, p_n \}$, with $p_0 = 0_{\mathscr{P}}$ and $p_n = 1_{\mathscr{P}}$. Let $\Phi = \{ \alpha_r : r \in J \}$ be a finite representation for $\mathscr{P}$, with $J = [0, d]$, $\forall r \leqslant d(\alpha_r : [0, n] \to [0, d])$, $\forall r \leqslant d(\alpha_r(0) = 0)$ and $\forall r_1, r_2 \leqslant d(r_1 \neq r_2 \to \alpha_{r_1}(n) \neq \alpha_{r_2}(n))$. Let

$$\{ D(\langle e, a \rangle) : e \in \omega, a \in [1, h] \}$$
$$\cup \{ W^a(x) : x \in \omega, a \in [1, h] \cup ([h + 1, m] \times \{1, 2\}) \}$$
$$\cup \{ CW(x) : x \in \omega \}$$

be a recursive partition of ω satisfying (15), (16) and $|W^a(x)| = 2^{x+1}$ for $x \in \omega$, $a \in [1, h] \cup ([h + 1, m] \times \{1, 2\})$. Let g be a 1-1 recursive map such that for each $x \in \omega$, $g \colon W(x) \times [h + 1, m] \times \{1, 2\} \to CW(x)$, and let $\{T_i \colon i \in \omega\}$ be an effective listing of the requirements P_e^a of (13) and Q_e^a of (14). For $a \in [1, h]$ let $r(a)_1$, $r(a)_2$ be recursive maps from $[1, h]$ to $[0, d]$ satisfying (19). For $(a, k) \in [h + 1, m]$ $\times \{1, 2\}$ let $r(a, k)_1$, $r(a, k)_2$ be recursive maps from $[h + 1, m] \times \{1, 2\}$ to $[0, d]$ satisfying (20). Our construction will define a recursive $f \colon \omega \times \omega \to [0, d]$. Let $A_{i,s} = \{ \langle x, \alpha_{f(x,s')}(i) \rangle \colon s' \leqslant s \}$, for $i \in [0, n]$, and $A_i = \bigcup_{s \in \omega} A_{i,s}$.

We define restraint functions $p(i, s)$, $q(i, s)$, and $R(i, s)$ by

$$p(i, s) = \begin{cases} \left| [e](\langle y, \alpha_{r(a)_2}(i_a) \rangle) \right| & \text{if } T_i = P_e^a \text{ and } \exists y \in D(\langle e, a \rangle) \\ & f(y, s) = r(a)_2 \text{ and} \\ & [e]^{A_{j_a,s}}(\langle y, \alpha_{r(a)_2}(i_a) \rangle) \downarrow = 0, \\ \uparrow & \text{ow,} \end{cases}$$

$$q(i, s) = \begin{cases} \left| [e](y) \right| & \text{if } T_i = Q_e^a \text{ and} \\ & \{e\}_s(y') \downarrow \text{ for all } y' \leqslant y \text{ and} \\ & [e]^{A_{i_a,s}} \upharpoonright y = [e]^{A_{j_a,s}} \upharpoonright y \text{ and} \\ & [e]^{A_{i_a,s}}(y) \neq [e]^{A_{j_a,s}}(y), \\ \uparrow & \text{ow,} \end{cases}$$

$R(i, s) = \max(\{\, p(j, s) \colon p(j, s)\downarrow \text{ and } j \leqslant i\,\} \cup \{\, q(j, s) \colon q(j, s)\downarrow \text{ and } j \leqslant i\,\})$.
If $\sigma \in d^{<\omega}$, then $A_{i, s+1}(\sigma)$ is the set that would be $A_{i, s+1}$ if we set $f(x, s + 1) = \sigma(x)$ for $x < |\sigma|$ and $f(x, s + 1) = f(x, s)$ for $x \geqslant |\sigma|$.

We say that "σ is a legal move for T_i at $s + 1 = 2t$" if (21)–(24) below hold.

(21) $\sigma \in d^{<\omega}$.

(22)   For all $x < |\sigma|(\sigma(x) \neq f(x, s) \Rightarrow x > R(i, s)$ and $f(x, s) = f(x, 0))$.

(23)   If $T_i = P_e^a$, then there is at most one x such that $\sigma(x) \neq f(x, s)$, and for this x, $x \in D(\langle e, a\rangle)$ and $\sigma(x) = r(a)_2$. (This clause says that if P_e^a acts, then it only changes f on one number, and this number is from the appropriate part of the partition of ω.)

(24)   If $T_i = Q_e^a$, then

(24.1) $(\forall x < |\sigma|)(\forall e' \in \omega)(\forall a' \in [1, h])(\sigma(x) \neq f(x, s)$ and $x \in D(\langle e', a'\rangle) \Rightarrow P_{e'}^{a'} = T_j$ and $j > i)$. (This clause says that Q_e^a can only touch the witnesses attached to positive requirements of lower priority than itself.)

(24.2) Let $B(x, b, \sigma, s) = \{\, y \in W^b(x) \colon \sigma(y) \neq f(y, s)$ or $\exists(a', k') \in [h + 1, m] \times \{1, 2\}(\sigma(g(y, a', k')) \neq f(g(y, a', k'), s))\}$. Then for all $x \in \omega$, $b \in ([1, h]) \cup ([h + 1, m] \times \{1, 2\})$

            if $x \in K_t$, then $B(x, b, \sigma, s) = \varnothing$,

            if $x \leqslant i$, then $B(x, b, \sigma, s) = \varnothing$,

            if x is frozen for T_i at s, then $B(x, b, \sigma, s) = \varnothing$,

            otherwise $|B(x, b, \sigma, s)| \leqslant 2^{x-i}$.

(This clause ensures that Q_e^a does not use up too many of the witnesses in the sets $W(x)$ for $x \in \omega$. This will ensure that there will always be enough witnesses available for coding $K(x)$.)

(24.3) For all x, $\exists y \in W(x)(\sigma(y) \neq f(y, s) \Rightarrow \exists(a', k') \in [h + 1, m] \times \{1, 2\}(\sigma(g(y, a', k')) \neq f(g(y, a', k'), 0))$. (This clause ensures that our decoding procedure is valid; if Q_e^a changes f on an element of $W(x)$, then it must also change f on a partner to that element.)

Next we define the phrase "T_i requires attention at s".

            $T_i = Q_e^a$ requires attention at $s + 1 = 2t + 1$ if $q(i, s)\downarrow$ and $k_t < q(i, s)$.

            $T_i = P_e^a$ requires attention at $s + 1 = 2t + 1$ if $p(i, s)\downarrow$ and $k_t < p(i, s)$.

            $T_i = Q_e^a$ requires attention at $s + 1 = 2t$ if $q(i, s)\uparrow$ and $\exists \sigma \in d^{<\omega}$, σ is a legal move for T_i at $s + 1$ and $\exists y(\{e\}_{s+1}(y')\downarrow$ for all $y' \leqslant y$ and $[e]^{A_{i,s}(\sigma)} \restriction y = [e]^{A_{j,s}(\sigma)} \restriction y$ and $[e]^{A_{i,s}(\sigma)}(y) \neq [e]^{A_{j,s}(\sigma)}(y))$.

            $T_i = P_e^a$ requires attention at $s + 1 = 2t$ if $p(i, s)\uparrow$ and there is a $y \in D(\langle e, a\rangle)$ such that $y > R(i, s)$ and $[e]^{A_{j,s}}(\langle y, \alpha_{r(a)_2}(i_a)\rangle)\downarrow = 0$ and $f(y, s) = f(y, 0) = r(a)_1$. (Let σ be the corresponding legal move.)

We now give the construction of f.

STAGE $s = 0$. For all $e \in \omega$, $a \in [1, h]$, for all $y \in D(\langle e, a \rangle)$, define $f(y, 0) = r(a)_1$. For all $x \in \omega$, $a \in [1, h]$, for all $y \in W^a(x)$, define $f(y, 0) = r(a)_1$. For all $x \in \omega$, $(a, k) \in [h + 1, m] \times \{1, 2\}$, for all $y \in W^{(a,k)}(x)$, for all $z \in W(x)$, define $f(y, 0) = r(a, k)_1$, and $f(g(z, a, k), 0) = r(a, k)_1$. Since

$$\left(\bigcup_{x \in \omega} W(x) \right) \cup \left(\bigcup_{x \in \omega} CW(x) \right) \cup \left(\bigcup_{e \in \omega} \left(\bigcup_{a \in [1, h]} D(\langle e, a \rangle) \right) \right) = \omega,$$

we have defined $f(y, 0)$ for all $y \in \omega$.

STAGE $s + 1 = 2t + 1$. $(t \geqslant 0)$ Let i be the least such that T_i requires attention at $2t + 1$. We say that T_i is active at $s + 1$.

If $T_i = P_e^a$, then let y be the least element, if any, of $W^a(k_t)$ such that $f(y, s) = f(y, 0)$ and for all $(a', k) \in [h + 1, m] \times \{1, 2\}$, $f(g(y, a', k), s) = f(g(y, a', k), 0)$. Define $f(y, s + 1) = r(a)_2$ and $f(y', s + 1) = f(y', s)$ for all $y' \neq y$. If no such y exists, then define $f(x, s + 1) = f(x, s)$ for all $x \in \omega$.

If $T_i = Q_e^a$, then let y_1, y_2 be the least elements, if any, of $W^{(a,1)}(k_t)$, $W^{(a,2)}(k_t)$, respectively, such that $f(y_1, s) = f(y_1, 0)$, $f(y_2, s) = f(y_2, 0)$, and for all $(a', k) \in [h + 1, m] \times \{1, 2\}$, $f(g(y_1, a', k), s) = f(g(y_1, a', k), 0)$ and $f(g(y_2, a', k), s) = f(g(y_2, a', k), 0)$. Let z be such that for all $z' \leqslant z(\{e\}_s(z')\downarrow)$ and $[e]^{A_{i_a, s}} \upharpoonright z = [e]^{A_{j_a, s}} \upharpoonright z$ and $[e]^{A_{i_a, s}}(z) \neq [e]^{A_{j_a, s}}(z)$.

Let $\sigma_1, \sigma_2, \sigma_3 \in d^{<\omega}$ have length $1 + \max\{y_1, y_2\}$ and be defined by

(25) $\sigma_1(y_1) = r(a, 1)_2$, $\sigma_1(x) = f(x, s)$ for all $x \neq y_1$, $x < |\sigma_1|$,

(26) $\sigma_2(y_2) = r(a, 2)_2$, $\sigma_2(x) = f(x, s)$ for all $x \neq y_2$, $x < |\sigma_2|$,

(27) $\sigma_3(y_1) = r(a, 1)_2$, $\sigma_3(y_2) = r(a, 2)_2$, $\sigma_3(x) = f(x, s)$

for all $x \neq y_1$, y_2, $x < |\sigma_3|$.

Choose $\tau = \sigma_1$, σ_2 or σ_3 so that

(28) $[e]^{A_{i_a, s+1}(\tau)}(z) \neq [e]^{A_{j_a, s+1}(\tau)}(z)$,

and define $f(x, s + 1) = \tau(x)$ for all $x < |\tau|$, $f(x, s + 1) = f(x, s)$ for all $x \geqslant |\tau|$. (Lemma 1 will show that one of σ_1, σ_2, σ_3 must satisfy (28).)

If there are no such y_1, y_2, then define $f(x, s + 1) = f(x, s)$ for all $x \in \omega$. If no T_i requires attention at $s + 1$, then let $y \in W(k_t)$ be the least such that y is reserved for $a = 1$ and $f(y, s) = f(y, 0)$ and for all $(a', k) \in [h + 1, m] \times \{1, 2\}$, $f(g(y, a', k), s) = f(g(y, a', k), 0)$. Define $f(y, s + 1) = r(1)_2$, and $f(x, s + 1) = f(x, s)$ for all $x \neq y$.

STAGE $s + 1 = 2t$. $(t \geqslant 1)$ Let i be the least such that T_i requires attention at $s + 1 = 2t$. We say that T_i is active at $s + 1$. If there is no such T_i, then go on to stage $s + 2$. Otherwise, let σ be the associated legal move. Define $f(y, s + 1) = \sigma(y)$ for all $y < |\sigma|$, and $f(y, s + 1) = f(y, s)$ for all $y \geqslant |\sigma|$. If $y < |\sigma|$ and $\sigma(y) \neq f(y, s)$ and $y \in W(x) \cup CW(x)$, then "x is frozen for T_j at s'", for all $s' \geqslant s + 1$ and all $j \in [i, x]$.

This ends the construction of f.

LEMMA 1. *If* $[e]^{A_{i_a,s}}(z)\downarrow \ne [e]^{A_{j_a,s}}(z)\downarrow$ *and* σ_1, σ_2 *and* σ_3 *are defined by* (25), (26) *and* (27), *then* (28) *holds for* $\tau =$ *one of* σ_1, σ_2, σ_3.

PROOF. Suppose (28) fails for σ_1 and σ_2, and that $b = [e]^{A_{i_a,s}}(z)$ and $1 - b = [e]^{A_{j_a,s}}(z)$. By definition, $\alpha_{r(a,1)_i}(i_a) = \alpha_{r(a,1)_2}(i_a)$ and $\alpha_{r(a,2)_1}(j_a) = \alpha_{r(a,2)_2}(j_a)$. So, $A_{i_a,s}(\sigma_1) = A_{i_a,s}$, and $A_{j_a,s}(\sigma_2) = A_{j_a,s}$, and $A_{i_a,s}(\sigma_3) = A_{i_a,s}(\sigma_2)$, and $A_{j_a,s}(\sigma_3) = A_{j_a,s}(\sigma_1)$. Since (28) fails for σ_1, we get

$$b = [e]^{A_{i_a,s}}(z) = [e]^{A_{i_a,s}(\sigma_1)}(z) = [e]^{A_{j_a,s}(\sigma_1)}(z)$$
$$= [e]^{A_{j_a,s}(\sigma_3)}(z).$$

Since (28) fails for σ_2, we get

$$1 - b = [e]^{A_{j_a,s}}(z) = [e]^{A_{j_a,s}(\sigma_2)}(z) = [e]^{A_{i_a,s}(\sigma_2)}(z)$$
$$= [e]^{A_{i_a,s}(\sigma_3)}(z).$$

and therefore $[e]^{A_{i_a,s}(\sigma_3)}(z) \ne [e]^{A_{j_a,s}(\sigma_3)}(z)$, as desired.

LEMMA 2. (a) A_0 *is recursive.*
(b) *For all* $i, j \leqslant n$, $p_i \leqslant p_j \Rightarrow A_i \leqslant_{\text{tt}} A_j$.
(c) *For all* $i, j, k \leqslant n$, $p_i \vee p_j = p_k \Rightarrow A_k \leqslant_{\text{tt}} A_i \oplus A_j$.

PROOF. As described in the outline before the construction.

LEMMA 3. *For each* $i \in \omega$, T_i *is active at most finitely often, and* $\lim_s R(i, s)$ *exists.*

PROOF. By induction on i. Let s_0 be such that $(\forall j < i)(\forall s \geqslant s_0)(T_j$ is not active at s and $R(i - 1, s) = R(i - 1, s_0))$. Suppose first that $T_i = P_e^a$. If $\exists s > s_0(p(i, s)\downarrow)$, then let s_1 be the least such. Suppose $s > s_1$, $s = 2t$, and T_j is active at s. Then $j \geqslant i$, since $s > s_1 \geqslant s_0$. Using induction on $s > s_1$, we see that $\forall s > s_1(p(i, s)\downarrow)$, and so by definition of T_i requires attention, we get that $j > i$. Now the definition of a legal move guarantees that $A_{j_a,s-1} \restriction p(i, s - 1) = A_{j_a,s} \restriction p(i, s - 1)$. If $s > s_1$, $s = 2t + 1$, and T_j is active at s, then $j \geqslant i$ (since $s > s_1 > s_0$) and the construction guarantees $A_{j_a,s} \restriction p(i, s - 1) = A_{j,s-1} \restriction p(i, s - 1)$. If $j = i$, then $k_t < p(i, s - 1)$. By induction on s we see that $\forall s \geqslant s_1(p(i, s) = p(i, s_1))$. Now if $s_2 \geqslant s_1$ is such that $K_{s_2} \restriction p(i, s_1) + 1 = K \restriction p(i, s_1) + 1$, then T_i will never be active after s_2.

Now suppose that $T_i = Q_e^a$. If $\exists s \geqslant s_0(q(i, s)\downarrow)$ then let s_1 be the least such. Then as above, $\forall s \geqslant s_1(q(i, s)\downarrow = q(i, s_1))$. If $s_2 \geqslant s_1$ is such that $K_{s_2} \restriction q(i, s_1) + 1 = K \restriction q(i, s_1) + 1$, then T_i will never be active after s_2.

Since for each i, $\exists s_0(\forall s \geqslant s_0 p(i, s)\uparrow$ or $\forall s \geqslant s_0(p(i, s)\downarrow = p(i, s_0))$ and $(\forall s \geqslant s_0(q(i, s)\uparrow)$ or $\forall s \geqslant s_0(q(i, s)\downarrow = q(i, s_0)))$, we get that $\lim_s R(i, s)$ exists.

LEMMA 4. *For all* $a \in [1, h]$, *for all* $e \in \omega$, P_e^a *is satisfied.*

PROOF. Let i be such that $T_i = P_e^a$. Let s_0 be such that $\forall s \geqslant s_0(R(i, s) = R(i, s_0))$ and $\forall j \leqslant i(T_j$ is not active at $s))$. Such a stage exists by Lemma 3. Suppose P_e^a fails, then $A_{i_a} = [e]^{A_{j_a}}$. Since for $y \in D(\langle e, a \rangle)$, $f(y, s + 1) \ne f(y, s)$ implies some T_j, $j \leqslant i$ was active at $s + 1$ and by Lemma 3 each T_j is active at most finitely often, there are at most finitely many $y \in D(\langle e, a \rangle)$ such that $\exists s \in \omega(f(y, s) \ne f(y, 0))$.

Let y be the least element of $D(\langle e, a \rangle)$ such that $y > R(i, s_0)$ and $\forall s \in \omega(f(y, s)$ $= f(y, 0))$. Then $\langle y, \alpha_{r(a)_2}(i_a) \rangle \notin A_{i_a}$, and since $A_{i_a} = [e]^{A_{j_a}}$, we get $[e]^{A_{j_a}}(\langle y, \alpha_{r(a)_2}(i_a) \rangle) = 0$. If $s_1 = 2t > s_0$ is such that

$$A_{j_a, s_1} \restriction \big| [e](\langle y, \alpha_{r(a)_2}(i_a) \rangle) \big| = A_{j_a} \restriction \big| [e](\langle y, \alpha_{r(a)_2} \rangle) \big|,$$

then T_i requires attention at s_1, and since no T_j, $j < i$ is active at s_1, T_i will become active at $s_1 > s_0$. This is a contradiction, so P_e^a must be satisfied.

LEMMA 5. $K \leqslant_{tt} A_n$.

PROOF. We show that

(29)
$$\forall x \in \omega \Big(x \in K \leftrightarrow \Big(\exists a \in [1, h] \exists y \in W^a(x) \big(\langle y, \alpha_{r(a)_2}(n) \rangle \in A_n \text{ and}$$

$$\forall (a', k') \in [h + 1, m] \times \{1, 2\} \forall r \leqslant d \big(r \neq r(a', k')_1 \rightarrow$$

$$\langle g(y, a', k'), \alpha_r(n) \rangle \notin A_n \big) \big) \Big)$$

or

$$\Big(\exists (a, k) \in [h + 1, m] \times \{1, 2\} \exists y \in W^{(a,k)}(x) \big(\langle y, \alpha_{r(a,k)_2}(n) \rangle \in A_n \text{ and}$$

$$\forall (a', k') \in [h + 1, m] \times \{1, 2\} \forall r \leqslant d \big(r \neq r(a', k')_1 \rightarrow$$

$$\langle g(y, a', k'), \alpha_r(n) \rangle \notin A_n \big) \big) \Big) \Big).$$

This provides the desired tt-reduction. We first show that for all t, $W(k_t)$ contains enough elements to carry out the prescribed coding action at stage $2t + 1$ in the construction. We say that $y \in W(x)$ is "free for coding at $s + 1$" if $f(y, s) = f(y, 0)$ and for all $(a, k) \in [h + 1, m] \times \{1, 2\}$, $f(g(y, a, k), s) = f(g(y, a, k), 0)$.

CLAIM 5.1. If $x = k_t$ and $a \in [1, h]$, then there is a $y \in W^a(x)$ such that y is free for coding at $2t + 1$. If $(a, k) \in [h + 1, m] \times \{1, 2\}$, then there is a $y \in W^{(a, k)}(x)$ such that y is free for coding at $2t + 1$.

PROOF. For $a \in [1, h]$ we define

$$V^a(x, s) = \big\{ y \in W^a(x) : f(y, s) \neq f(y, 0) \text{ or } \exists (a', k') \in [h + 1, m] \times \{1, 2\}$$

$$f(g(y, a', k'), s) \neq f(g(y, a', k'), 0) \big\}.$$

We must show that there is a $y \in W^a(x)$ such that $y \notin V^a(x, 2t)$ (where $x = k_t$). Since $|W^a(x)| = 2^{x+1}$, it suffices to show $|V^a(x, 2t)| \leqslant 2^{x+1} - 2$. Note that $V^a(x, 2t) = \bigcup_{s' < 2t} (V^a(x, s' + 1) - V^a(x, s'))$. Also note that if $s' + 1 = 2t' + 1 \leqslant 2t$, then $k_{t'} \neq x$ (the enumeration is 1-1) and so our construction ensures

$$V^a(x, s' + 1) - V^a(x, s') = \varnothing.$$

If $s' + 1 = 2t' \leqslant 2t$ and $V^a(x, s' + 1) - V^a(x, s') \neq \varnothing$, then some $T_i = Q_e^{a'}$ was active at $s' + 1$, and executed a legal move. Since $V^a(x, s' + 1) - V^a(x, s') \neq \varnothing$, the definition of a legal move guarantees that $i < x$, and $|V^a(x, s' + 1) - V^a(x, s')| \leqslant 2^{x-i}$. The process of freezing x for T_i guarantees that for each i there is at most

one s' such that (T_i is active at $s' + 1$ and $V^a(x, s' + 1) - V^a(x, s') \neq \varnothing$). Therefore,

$$|V^a(x, 2t)| \leqslant \sum_{i < x} 2^{x-i} = 2^{x+1} - 2.$$

If we replace a everywhere by $(a, k) \in [h + 1, m] \times \{1, 2\}$, then the above argument shows that there is a $y \in W^{(a,k)}(k_t)$ which is free for coding at $2t + 1$.

The construction guarantees that if $x \in K_t$, then for all $s \geqslant 2t + 2$, for all $y \in W(x) \cup CW(x)$, $f(y, s + 1) = f(y, s)$. This, together with claim 5.1 establishes " $\rightarrow$ " for (29). We establish " $\leftarrow$ " for (29) by showing

CLAIM 5.2. *If* $s + 1 = 2t$ *and* $x \notin K_t$ *and if* $\exists y \in W(x)(f(y, s + 1) \neq f(y, s))$, *then*

$$\exists(a', k') \in [h + 1, m] \times \{1, 2\}(f(g(y, a', k'), s + 1) \neq f(g(y, a', k'), 0)).$$

PROOF. If $y \in W(x)$ and $f(y, s + 1) \neq f(y, s)$, then some $T_i = Q_e^a$, with $i < x$ must have been active at $s + 1$ and executed a legal move. The definition of a legal move, in particular (24.3), guarantees that the claim holds.

Now suppose $x \notin K$. If $\forall a \in [1, h]$ $\forall y \in W^a(x)(\langle y, \alpha_{r(a)_2}(n) \rangle \notin A_n)$ and $\forall(a, k) \in [h + 1, m] \times \{1, 2\}$ $\forall y \in W^{(a,k)}(x)(\langle y, \alpha_{r(a,k)_2}(n) \rangle \notin A_n)$, then (29) is satisfied. Otherwise, if $a \in [1, h]$ and $y \in W^a(x)$ is such that $\langle y, \alpha_{r(a)_2}(n) \rangle \in A_n$, then since $\alpha_{f(y,0)}(n) \neq \alpha_{r(a)_2}(n)$, there is an $s + 1$ such that $f(y, s + 1) \neq f(y, s)$. Since $x \notin K$, $s = 2t$ and $x \notin K_t$, and so Claim 5.2 guarantees $\exists(a', k') \in [h + 1, m] \times \{1, 2\}$ $\exists r \in J(r \neq f(g(y, a', k'), 0))$ and $\langle g(y, a', k'), \alpha_r(n) \rangle \in A_n$. If we replace a everywhere by (a, k) in the above argument, then we have established (29).

LEMMA 6. *For all* $a \in [h + 1, m]$, *for all* $e \in \omega$, Q_e^a *is satisfied.*

PROOF. Suppose that $[e]^{A_{i_a}} = [e]^{A_{j_a}} = Y$. We want to show $Y \leqslant_{tt} A_{k_a}$. Let i be such that $T_i = Q_e^a$. Let s_0 be such that $\forall s \geqslant s_0$ $\forall j \leqslant i$ (T_j is not active at s and $R(i, s) = R(i, s_0) = R(i)$). Let

$$q = 1 + \max(\{i, R(i)\} \cup \{x : x \text{ is frozen for } T_i \text{ at } s_0\}).$$

Let $s_1 \geqslant s_0$ be such that $K_{s_1} \upharpoonright q = K \upharpoonright q$. In (30) we outline a reduction procedure from an arbitrary oracle, B. This reduction procedure will be total for all oracles, B, and so by Nerode's result [6, p. 143] we have a tt-reduction. We first outline the reduction procedure for input y and then verify that when $B = A_{k_a}$ the procedure computes Y.

(30)   Find a stage $s_2 = 2t \geqslant s_1$ such that for all $y' \leqslant y(\{e\}_{s_2}(y') \downarrow)$. See if there is a $\sigma \in d^{<\omega}$ such that

$$|\sigma| = |[e](y)| \text{ and } B \upharpoonright \langle |[e](y)|, d \rangle = A_{k_a, s_2}(\sigma) \upharpoonright \langle |[e](y)|, d \rangle,$$

and for all $z < |\sigma|$, if ($\forall r \in J(\langle z, \alpha_r(k_a) \rangle \in B \Rightarrow \alpha_r(k_a) = \alpha_{f(z,0)}(k_a)$ or $\alpha_r(k_a) = \alpha_{f(z,s_2)}(k_a))$) then $\sigma(z) = f(z, s_2)$. If so then output $[e]^{A_{i_a, s_2}(\sigma)}(y)$, if not then output y.

We now show that $[e]^{A_{i_a,s_2}(\sigma)}(y) = [e]^{A_{i_a}}(y) = Y(y)$, where σ is the string used in (30). Let $s_3 \geqslant s_2$ be such that $A_{i_a,s_3} \upharpoonright \langle |[e](y)|, d \rangle = A_{i_a} \upharpoonright \langle |[e](y)|, d \rangle$. Define $\tau \in d^{<\omega}$ so that $|\tau| = |[e](y)|$ and $\tau(x) = f(x, s_3)$ for all $x < |\tau|$. Then $[e]^{A_{i_a}}(y) = [e]^{A_{i_a,s_2}(\tau)}(y)$, so it suffices to show

(31)
$$[e]^{A_{i_a,s_2}(\sigma)}(y) = [e]^{A_{i_a,s_2}(\tau)}(y).$$

Notice that for all $x < |[e](y)| = |\sigma| = |\tau|$, $\alpha_{\sigma(x)}(k_a) = \alpha_{\tau(x)}(k_a)$. This is true since if $\alpha_{\sigma(x)}(k_a) \neq \alpha_{f(x,s_2)}(k_a)$ then the definition of σ in (30) guarantees that $f(x, s_2) = f(x, 0)$ and $\exists s > s_2(f(x, s) \neq f(x, s_2)$, and $\alpha_{f(x,s)}(k_a) \neq \alpha_{f(x,0)}(k_a))$. But since $p_{k_a} \leqslant p_{i_a}$, (1) implies that $\alpha_{f(x,s)}(i_a) \neq \alpha_{f(x,0)}(i_a)$, and so $s \leqslant s_3$ and so $f(x, s) = f(x, s_3) = \tau(x)$. The definition of σ also guarantees that if $\alpha_{\sigma(x)}(k_a) \neq \alpha_{f(x,s_2)}(k_a)$, then $\alpha_{\tau(x)}(k_a) = \alpha_{f(x,s_2)}(k_a) = \alpha_{\sigma(x)}(k_a)$.

Now since $\alpha_{\sigma(x)}(k_a) = \alpha_{\tau(x)}(k_a)$, (3) implies that for each $x < |\sigma|$ such that $\sigma(x) \neq \tau(x)$ we can find a sequence $r_{x_0}, \dots, r_{x_{m_x}}$ from J such that $r_{x_0} = \sigma(x)$, $r_{x_{m_x}} = \tau(x)$, and for $b < m_x - 1$, b even,

$$\alpha_{r_{x_b}}(i_a) = \alpha_{r_{x_{b+1}}}(i_a) \quad \text{and} \quad \alpha_{r_{x_{b+1}}}(j_a) = \alpha_{r_{x_{b+2}}}(j_a).$$

Let $p = \max\{m_x : \sigma(x) \neq \tau(x)\}$ and define $\sigma_0, \dots, \sigma_p \in d^{<\omega}$ of length $|\sigma|$ as follows for $b \in [0, p]$

$$\sigma_b(x) = \begin{cases} r_{x_b} & \text{if } b \leqslant m_x \text{ and } \sigma(x) \neq \tau(x), \\ r_{x_{m_x}} & \text{if } b \geqslant m_x \text{ and } \sigma(x) \neq \tau(x), \\ \sigma(x) & \text{if } \sigma(x) = \tau(x). \end{cases}$$

Now we have $\sigma_0 = \sigma$, $\sigma_p = \tau$, and for $b \in [0, p - 1]$, b even,

(32)
$$A_{i_a,s_2}(\sigma_b) \upharpoonright \langle |[e](y)|, d \rangle = A_{i_a,s_2}(\sigma_{b+1}) \upharpoonright \langle |[e](y)|, d \rangle$$

and

(33)
$$A_{j_a,s_2}(\sigma_{b+1}) \upharpoonright \langle |[e](y)|, d \rangle = A_{j_a,s_2}(\sigma_{b+2}) \upharpoonright \langle |[e](y)|, d \rangle.$$

If we can show, for $b \in [0, p]$

(34)
$$[e]^{A_{i_a,s_2}(\sigma_b)}(y) = [e]^{A_{j_a,s_2}(\sigma_b)}(y),$$

then by combining (32), (33) and (34) we get

$$[e]^{A_{i_a,s_2}(\sigma)}(y) = [e]^{A_{i_a,s_2}(\sigma_0)}(y) = [e]^{A_{i_a,s_2}(\sigma_1)}(y)$$
$$= [e]^{A_{j_a,s_2}(\sigma_1)}(y) = [e]^{A_{j_a,s_2}(\sigma_2)}(y) = [e]^{A_{i_a,s_2}(\sigma_2)}(y)$$
$$= \cdots = [e]^{A_{j_a,s_2}(\sigma_p)}(y) = [e]^{A_{i_a,s_2}(\sigma_p)}(y)$$
$$= [e]^{A_{i_a,s_2}(\tau)}(y).$$

The outside terms in this string of equalities give (31), and hence the lemma.

Suppose that (34) fails for some $b \in [0, p]$, and let b be the least such. We will construct a legal move for T_i at s_2, μ, from σ_b, such that $[e]^{A_{i_a,s_2}(\mu)}(y) \neq [e]^{A_{j_a,s_2}(\mu)}(y)$, and hence T_i will require attention and become active at s_2, a contradiction.

We obtain μ by patching up σ_b to satisfy (24.3). Let $\{\langle y_1, x_1 \rangle, \ldots, \langle y_u, x_u \rangle\}$ be a list of all the pairs $\langle y_i, x_i \rangle$ such that $y_i \in W(x_i)$ and $\sigma_b(y_i) \neq f(y_i, s_2)$ and for all $(a', k') \in [h + 1, m] \times \{1, 2\}$, $\sigma_b(g(y_i, a', k')) = f(g(y_i, a', k'), 0) = r(a', k')_1$. For each $i \leq u + 1$ we will define a σ_b^i so that $\sigma_b^0 = \sigma_b$ and for $i \leq u$

(35) $$\{z \colon \sigma_b^{i+1}(z) \neq \sigma_b^i(z)\} \subseteq \{g(y_i, a, k) \colon k \in \{1, 2\}\}$$

and

(36) $$[e]^{A_{i_a, s_2}(\sigma_b^{i+1})}(y) \neq [e]^{A_{j_a, s_2}(\sigma_b^{i+1})}(y).$$

Suppose we have defined σ_b^i satisfying (35) and (36). To get σ_b^{i+1}, let τ_1, τ_2, $\tau_3 \in d^{<\omega}$ have length $|\sigma_b|$ and be defined by

$$\tau_1(x) = \begin{cases} r(a, 1)_2 & \text{if } x = g(y_i, a, 1), \\ \sigma_b^i(x) & \text{ow} \end{cases}$$

$$\tau_2(x) = \begin{cases} r(a, 2)_2 & \text{if } x = g(y_i, a, 2), \\ \sigma_b^i(x) & \text{ow,} \end{cases}$$

$$\tau_3(x) = \begin{cases} r(a, 1)_2 & \text{if } x = g(y_i, a, 1), \\ r(a, 2)_2 & \text{if } x = g(y_i, a, 2), \\ \sigma_b^i(x) & \text{ow,} \end{cases}$$

then (35) holds for $\sigma_b^{i+1} = $ any of τ_1, τ_2 or τ_3, and just as in the proof of Lemma 1, (36) must hold for $\sigma_b^{i+1} = $ one of τ_1, τ_2 or τ_3. So we choose $\sigma_b^{i+1} = \tau_1$, τ_2 or τ_3 to make (36) hold.

Let $\mu = \sigma_b^{u+1}$. We show that μ is a legal move for T_i at s_2 by verifying (22)–(24) for μ, T_i and s_2.

CLAIM 6.1. For all $y < |\mu|$, if $\mu(y) = \sigma_b(y)$ and $\mu(y) \neq f(y, s_2)$, then there is an $s \geq s_2$ such that $f(y, s + 1) \neq f(y, s)$.

PROOF. Suppose $\mu(y) = \sigma_b(y)$ and $\mu(y) \neq f(y, s_2)$. If $\sigma(y) = \tau(y)$, then $\sigma_b(y) = \tau(y) = f(y, s_3)$, and $f(y, s_3) \neq f(y, s_2)$, so for some $s \geq s_2 f(y, s + 1) \neq f(y, s)$. As noted above, $\forall x < |\sigma|$, $\alpha_{\sigma(x)}(k_a) = \alpha_{\tau(x)}(k_a)$, so if $\sigma(y) \neq \tau(y)$, then $\sigma(y) = f(y, s_2)$. Thus we have $f(y, s_2) \neq f(y, s_3)$, so for some $s \geq s_2$ $f(y, s + 1) \neq f(y, s)$.

VERIFICATION OF (22)–(24).

(22) We need to show that if $\mu(y) \neq f(y, s_2)$, then $y > R(i, s_2)$ and $f(y, s_2) = f(y, 0)$. Suppose $\mu(y) \neq f(y, s_2)$. If $\mu(y) = \sigma_b(y)$, then Claim 6.1 guarantees that $\exists s \geq s_2$, $f(y, s + 1) \neq f(y, s)$. If $s + 1 = 2t$, then some T_j is active at $s + 1$, and since $s + 1 \geq s_2 > s_0$, j must be greater than i. T_j must have executed a legal move at $s + 1$, so (22) for T_j at $s + 1$ implies that $y > R(j, s + 1) \geq R(i, s + 1) = R(i, s_2)$. Also, if $f(y, s + 1) \neq f(y, s)$ then (22) for T_j at $s + 1$ implies that $f(y, s) = f(y, s_2) = f(y, 0)$. If $s + 1 = 2t + 1$, then since $K_{s_1} \restriction R(i) = K \restriction R(i)$, $k_t > R(i)$. Since $y \in W(k_t)$ and all elements of $W(k_t)$ are greater than k_t, we get $y > R(i) = R(i, s_2)$, and again $f(y, s) = f(y, s_2) = f(y, 0)$. Thus we get (22) for $\mu(y)$ if $\mu(y) = \sigma_b(y)$. If $\mu(y) \neq \sigma_b(y)$, then $y = g(z, a, k)$ for some $k \in \{1, 2\}$ and z such that $\sigma_b(z) \neq f(z, s_2)$. Since $\mu(z) = \sigma_b(z)$, the above argument shows that $z > R(i)$. Since $g(z, a, k) > z$, we get $y > z > R(i) = R(i, s_2)$. Our choice of the σ_b^i for $i \leq u + 1$ guarantees that $f(y, s_2) = f(y, 0)$, so we get (22) for $\mu(y) \neq \sigma_b(y)$.

(23) This condition applies only when $T_i = P_{e'}^{a'}$, and so is satisfied vacuously, since $T_i = Q_e^a$.

(24.1) We need to show that if $\mu(y) \neq f(y, s_2)$ and $y \in D(\langle e', a' \rangle)$, then $P_{e'}^{a'} = T_j$ with $j > i$. If $\mu(y) \neq f(y, s_2)$ and $y \in D(\langle e', a' \rangle)$, then $\mu(y) = \sigma_b(y)$, and by Claim 6.1, $\exists s \geq s_2 (f(y, s+1) \neq f(y, s))$. Since $y \in D(\langle e', a' \rangle)$, it must be the case that $s + 1 = 2t$ and some T_j is active at $s + 1$. Since $s + 1 > s_0$, $j > i$. So if $T_j = P_{e'}^{a'}$, then (24.1) holds for $\mu(y)$ and T_i at s_2. Otherwise, $T_j = Q_{e''}^{a''}$ for some e'', a'', and $P_{e'}^{a'} = T_k$. Then since T_j must have made a legal move at $s + 1$, (24.1) for T_j at $s + 1$ implies $k > j > 1$, so $k > i$ and (24.1) holds for $\mu(y)$ and T_i, at s_2.

(24.2) We need to show that the sets $B(x, c, \mu, s_2) = \{ y \in W^c(x) : \mu(y) \neq f(y, s_2)$ or $\exists (a', k') \in [h + 1, m] \times \{1, 2\}(\mu(g(y, a', k')) \neq f(g(y, a', k'), s_2))\}$ have the correct cardinalities. Fix $x \in \omega$ and $c \in [1, h] \cup [h + 1, m] \times \{1, 2\}$ and consider $B(x, c, \mu, s_2)$. If $y \in B(x, c, \mu, s_2) \neq \varnothing$, then either $\mu(y) = \sigma_b(y) \neq f(y, s_2)$ or $(\mu(y) = \sigma_b(y) = f(y, s_2)$ and $\mu(g(y, a', k')) \neq f(g(y, a', k'), s_2)$ for some $(a', k') \in [h + 1, m] \times \{1, 2\})$. If $\mu(y) = \sigma_b(y) = f(y, s_2)$, then our definition of μ guarantees that $\mu(g(y, a', k')) = \sigma_b(g(y, a', k'))$, and so we get that $\sigma_b(g(y, a', k')) \neq f(g(y, a', k'), s_2)$. Thus by Claim 6.1 we know that if $y \in B(x, c, \mu, s_2)$ then there must be an $s \geq s_2$ such that either $f(y, s + 1) \neq f(y, s)$ or $f(g(y, a'k'), s + 1) \neq f(g(y, a', k'), s)$. So one of (37) or (38) below must hold.

(37) $$\exists s > s_2 (s = 2t + 1 \text{ and } x = k_t), \text{ or}$$

(38) $$\exists s > s_2 \exists j < x \exists \eta \in d^{<\omega}(s = 2t \text{ and } \eta \text{ is a legal move}$$
$$\text{for } T_j \text{ at } s \text{ and } y \in B(x, c, \eta, s)).$$

Condition (37) can occur at most once, and can contribute at most one element to $B(x, c, \mu, s_2)$. Condition (38) can occur at most once for each $j < x$, and so for a fixed $j < x$ can contribute at most 2^{x-j} elements to $B(x, c, \mu, s_2)$. Since $s_2 \geq s_0$, no T_k with $k \leq i$ can be active at $s \geq s_2$, so we need only consider T_j such that $i < j < x$, and we get

$$|B(x, c, \mu, s_2)| \leq 1 + \sum_{i < j < x} 2^{x-j} = 1 + (2^{x-i} - 2) \leq 2^{x-i}.$$

Also notice that if $s_2 = 2t$ and $x \in K_t$, then for all $y \in W(x) \cup CW(x)$, for all $s \geq s_2$, $f(y, s) = f(y, s_2)$, and so $\mu(y) = \sigma_b(y) = f(y, s_2)$, and so $B(x, c, \mu, s_2) = \varnothing$. If $x \leq i$, then since $K_{s_2} \upharpoonright i = K \upharpoonright i$, and no T_j, $j \leq i$ is active at $s \geq s_2$, again we get that $\forall y \in W(x) \cup CW(x) \forall s \geq s_2(f(y, s) = f(y, s_2))$ and so $B(x, c, \mu, s_2) = \varnothing$. If x is frozen for T_i at s_2, let $s' \leq s_2$ be the least such that x first became frozen for T_i at s'. Then some T_j, $j \leq i$ was active at s', and so $s' < s_0$. But then $x < q$, and so $K_{s_2} \upharpoonright x + 1 = K \upharpoonright x + 1$. Since x is frozen for T_i at s_2, x is frozen for T_k at s, for all $s \geq s_2$, for all k ($i \leq k < x$). Therefore for all $y \in W(x) \cup CW(x)$, for all $s \geq s_2$, $f(y, s + 1) = f(y, s)$, and so $B(x, c, \mu, s_2) = \varnothing$. Thus we have shown that (24.2) holds for μ and T_i at s_0.

(24.3) We need to show for any x that if there is a $y \in W(x)$ such that $\mu(y) \neq f(y, s_2)$, then there is a pair $(a', k') \in [h + 1, m] \times \{1, 2\}$ such that $\mu(g(y, a', k')) \neq f(g(y, a', k')), s_2$. Our definition of μ as a modification of σ_b ensures that (24.3) holds.

Thus we have shown that μ is a legal move for T_i at s_2, and that $[e]^{A_{i_a,s_2}(\mu)}(y) \neq [e]^{A_{j_a,s_2}(\mu)}(y)$, so T_i becomes active at s_2, a contradiction. We conclude that Q_e^a is satisfied.

COROLLARY 1. *Every finite lattice with more than one element can be embedded into the r.e. tt-degrees preserving least and greatest element.*

PROOF. The corollary follows from Theorem 1 since every finite lattice has a finite representation [5].

2. Recursively presentable lattices. In this section we will modify the proof of Theorem 1 to show that every recursively presentable lattice can be embedded in the r.e. tt-degrees preserving least and greatest elements. Note that by Shore [7] not every lattice can be embedded in this way; some restriction on the degree of the lattice presentation is necessary.

THEOREM 2. *Every lattice, $\mathscr{P}$, which has a recursive presentation and distinct least and greatest elements can be embedded into the r.e. tt-degrees preserving least and greatest elements.*

PROOF. We fix some notation first. Let $\mathscr{P} = \langle P, \leqslant, \vee, \wedge, 0_{\mathscr{P}}, 1_{\mathscr{P}} \rangle$ be a lattice with $0_{\mathscr{P}} \neq 1_{\mathscr{P}}$. Let $\Phi = \{\alpha_r : r \in \omega\}$ be a recursive representation of $\mathscr{P}$. Since $\mathscr{P}$ has a recursive presentation, we know that such a representation exists. Let $P = \{p_i : i \in \omega\}$, with $p_0 = 0_{\mathscr{P}}$, and $p_1 = 1_{\mathscr{P}}$. We assume, w.l.o.g., that $\forall r \in \omega, \alpha_r(0) = 0$ and $\forall r_1, r_2 \in \omega(r_1 \neq r_2 \Rightarrow \alpha_{r_1}(1) \neq \alpha_{r_2}(1))$. Let $\{R_a : a \in \omega\}$ be an effective listing of all the relations "$p_i \nleqslant p_j$" or "$p_i \wedge p_j = p_k$" in P, i.e. R_a is "$p_{i_a} \nleqslant p_{j_a}$" or "$p_{i_a} \wedge p_{j_a} = p_{k_a}$", and if $p_i \nleqslant p_j$ then $\exists a \in \omega, R_a =$"$p_i \nleqslant p_j$", and if $p_i | p_j$ and $p_i \wedge p_j = p_k$, then $\exists a \in \omega, R_a =$"$p_i \wedge p_j = p_k$". Define

$$H_x = \{a < x: R_a = "p_{i_a} \nleqslant p_{j_a}"\}, \qquad h_x = |H_x|,$$

$$M_x = \{a < x: R_a = "p_{i_a} \wedge p_{j_a} = p_{k_a}"\}, \qquad m_x = |M_x|.$$

For $a \in \bigcup_{x \in \omega} H_x$, define P_e^a as in (13), and for $a \in \bigcup_{x \in \omega} M_x$, define Q_e^a as in (14). Let $\{T_i : i \in \omega\}$ be an effective listing of the requirements $\{P_e^a : e \in \omega \text{ and } a \in \bigcup_{x \in \omega} H_x\}$ and $\{Q_e^a : e \in \omega \text{ and } a \in \bigcup_{x \in \omega} M_x\}$. As in the proof of Theorem 1, we construct a recursive function $f: \omega \times \omega \to \omega$ and for $i \in \omega$ define $A_{i,s} = \{\langle x, \alpha_{f(x,s')}(i) \rangle : x \in \omega \text{ and } s' \leqslant s\}$, and $A_i = \bigcup_s A_{i,s}$. The proof of Theorem 1 is modified by first changing the definition of a legal move, and second changing the strategy for cooperation between coding and the $\{T_i : i \in \omega\}$.

We modify the legal move definition to get (5) and (6). Since the range of f is now infinite, (5) and (6) are no longer automatic, but they will be if we establish

(39)   There is a recursive function F such that for $x, s \in \omega, f(x, s) \leqslant F(x)$.

As in Fejer and Shore [1], to establish (39) we define sets $S(i, y)$ for $i, y \in \omega$, where $S(i, y)$ represents T_i's guess at the possible values for $f(y, s)$. First we define

$J(i, r_1, r_2)$ for $i, r_1, r_2 \in \omega$ by

$$
J(i, r_1, r_2) = \begin{cases}
\varnothing & \text{if } \exists a \in \left(\bigcup_{x \in \omega} H_x \right) \exists e \in \omega \, (T_i = P_e^a), \\[2ex]
\varnothing & \text{if } \exists a \in \left(\bigcup_{x \in \omega} M_x \right) \exists e \in \omega \, (T_i = Q_e^a) \text{ and} \\[1ex]
& \alpha_{r_1}(k_a) \neq \alpha_{r_2}(k_a), \\[2ex]
\{s_0, \ldots, s_m\} & \text{if } \exists a \in \bigcup_{x \in \omega} M_x \; \exists e \in \omega \, (T_i = Q_e^a) \text{ and} \\[1ex]
& \alpha_{r_1}(k_a) = \alpha_{r_2}(k_a), \text{ and } s_0, \ldots, s_m \\[1ex]
& \text{satisfy (3) for } p_{i_a} \wedge p_{j_a} = p_{k_a} \\[1ex]
& \text{and } r_1, r_2. \text{ (Choose one fixed} \\[1ex]
& \text{set of interpolants for each} \\[1ex]
& \text{such triple } (a, r_1, r_2).)
\end{cases}
$$

We modify our recursive partition of ω so that we get

$$
\left\{ D(\langle e, a \rangle) : e \in \omega, a \in \bigcup_{x \in \omega} H_x \right\} \cup \left\{ W^a(x) : x \in \omega, a \in H_x \right\}
$$

$$
\cup \left\{ W^{(a,k)}(x) : x \in \omega, (a, k) \in M_x \times \{1, 2\} \right\} \cup \left\{ CW(x) : x \in \omega \right\}
$$

satisfying (15) and

(16′) Each $x \in \omega$ has two finite sets of numbers associated with it, call them $W(x)$ and $CW(x)$. Each element of $W(x) \cup CW(x)$ is greater than x, and $\min(CW(x)) > \max(W(x))$. Partition $W(x)$ into sets $W^a(x)$ for $a \in H_x$ and $W^{(a,k)}(x)$ for $(a, k) \in M_x \times \{1, 2\}$ so that $|W^a(x)| = 2^{x+1}$ for $a \in H_x$, $x \in \omega$ and $|W^{(a,k)}(x)| = 2^{x+1}$ for $(a, k) \in M_x \times \{1, 2\}$, $x \in \omega$ and $|CW(x)| = 2 \cdot m_x \cdot 2^{x+1}(h_x + 2m_x)$.

As before, g will be a 1-1 recursive function such that for all x, $g: W(x) \times M_x \times \{1, 2\} \to CW(x)$. For $a \in \bigcup_{x \in \omega} H_x$, $r(a)_1, r(a)_2$ are defined to satisfy (19), and for $(a, k) \in \bigcup_{x \in \omega} M_x \times \{1, 2\}$, $r(a, k)_1, r(a, k)_2$ are defined to satisfy (20). We are now ready to define the $S(i, y)$. For $i \geq y + 1$, define

$$
S(i, y) = \begin{cases}
\{r(a), r(a)_2\} & \text{if } a \in \bigcup_{x \in \omega} H_x \text{ and} \\[1ex]
& y \in \left(\bigcup_{e \in \omega} D(\langle e, a \rangle) \right) \cup \left(\bigcup_{x \in \omega} W^a(x) \right), \\[2ex]
\{r(a, k)_1, r(a, k)_2\} & \text{if } (a, k) \in \bigcup_{x \in \omega} M_x \times \{1, 2\} \\[1ex]
& \text{and } \exists z \in \bigcup_{x \in \omega} W(x), \; y = g(z, a, k), \\[2ex]
\{r(a, k)_1, r(a, k)_2\} & \text{if } (a, k) \in \bigcup_{x \in \omega} M_x \times \{1, 2\} \\[1ex]
& \text{and } y \in \bigcup_{x \in \omega} W^{(a,k)}(x).
\end{cases}
$$

Given $S(i + 1, y)$ with $0 \leqslant i \leqslant y$, define

$$S(i, y) = S(i + 1, y) \cup \bigcup \{ J(i, r_1, r_2) : r_1, r_2 \in S(i + 1, y) \}.$$

We modify the definition of "σ is a legal move for T_i at $s + 1$" by replacing (22) with

(22')   For $x < |\sigma|$ $(\sigma(x) \neq f(x, s) \rightarrow x > R(i, s)$ and $x > i$ and
$f(x, s) = f(x, 0)$ and $\sigma(x) \in S(i, x))$.

Now for all y, if for some s, $f(y, s + 1) \neq f(y, 0)$, then this definition of a legal move ensures that $f(y, s + 1) \in S(0, y)$. Hence if we define $F(x) = \max(S(0, x))$, then the new construction will satisfy (39). The proof of the analog of Lemma 6 for the new construction remains basically the same as the original proof. When considering $T_i = Q_e^a$, the legal move μ at stage s_2 constructed in the proof is such that if $\mu(y) \neq f(y, s_2)$, then $y = g(z, a', k')$ for some z, a', k', or $\exists s' \geqslant s_2$, $f(y, s' + 1) \neq f(y, s')$. If $y = g(z, a', k')$, then $\mu(y) = r(a', k')_2 \in S(i, y)$. If $\exists s' \geqslant s_2(f(y, s' + 1) \neq f(y, s'))$, then some T_j, $j > i$, was active at $s' + 1$, and $f(y, s')$, $f(y, s' + 1) \in S(j, y) \subseteq S(i + 1, y)$. We constructed μ so that $\mu(y) =$ one of the interpolants for $\alpha_{f(y, s')}(k_a) = \alpha_{f(y, s' + 1)}(k_a)$, and so $\mu(y) \in S(i, y)$. Hence the μ constructed will satisfy the new definition of legal move, and the proof of Lemma 6 is as before.

The coding strategy is modified as follows. The list of relations $\{ R_a : a \in \omega \}$ is infinite now, and so our coding action for x cannot cooperate with all of the R_a. Instead we insist that the action for coding x cooperate with R_a, only for $a < x$. Thus our action for coding the fact "$x \in K$" is chosen from among $h_x + 3m_x$ many possibilities, and we still get a tt-reduction for K from A_1. We establish

(29') $\forall x \in \omega \big(x \in K \leftrightarrow \exists a \in H_x \exists y \in W^a(x) \big(\langle y, \alpha_{r(a)_2} \rangle \in A_1$ and

$$\forall (a', k') \in M_x \times \{1, 2\} \forall r \leqslant F(g(y, a', k'))$$

$$\langle r \neq r(a', k')_1 \rightarrow \langle g(y, a', k'), \alpha_r(1) \rangle \notin A_1 \rangle \big) \big)$$

or

$\exists (a, k) \in M_x \times \{1, 2\} \exists y \in W^{(a, k)}(x)$

$\langle y, \alpha_{r(a, k)_2} \rangle \in A_1$ and $\forall (a', k') \in M_x \times \{1, 2\}$

$\forall r \leqslant F(g(y, a', k'))(r \neq r(a', k')_1 \rightarrow \langle g(y, a', k'), \alpha_r(1) \rangle \notin A_1)$.

The proofs of the lemmas showing the correctness of the construction will be as before, except that when considering $T_i = Q_e^a$ or P_e^a, we start from a stage s_0 such that $K_{s_0} \upharpoonright a = K \upharpoonright a$; then for $s \geqslant s_0$, if T_i has priority at s, then T_i will not be injured by coding at s. Thus Lemmas 3, 4, 5 and 6 will hold if we replace $[1, h]$ by H_x or $\bigcup_{x \in \omega} H_x$, $[h + 1, m]$ by M_x or $\bigcup_{x \in \omega} M_x$ whenever appropriate and let s_0 be large enough so that $K_{s_0} \upharpoonright a = K \upharpoonright a$.

COROLLARY 2. *The $\exists$ theory of the r.e. tt-degrees in the language $\mathscr{L}$ with $\leqslant$, $\vee$, $\wedge$ as relation symbols and $0, 1$ as constant symbols is decidable.*

PROOF. The decision procedure is the same as the one used in Fejer and Shore [1]. An existential sentence $\sigma = \exists x_1, \ldots, \exists x_n \varphi$ in $\mathscr{L}$ will be valid in the r.e. tt-degrees iff it is valid in some partial order with distinct least and greatest elements and at most

$(n + 2) + 2(n + 2)^3$ elements. Suppose σ is valid in some finite partial order $\mathscr{P}$ with distinct least and greatest elements. $\mathscr{P}$ can be extended to a finite lattice, $\mathscr{P}'$, with distinct least and greatest elements. Since $\mathscr{P}'$ is finite, it is recursively presentable, and so by Theorem 2 $\mathscr{P}'$ can be embedded in the r.e. tt-degrees. The embedding preserves all joins and infs mentioned in φ, as well as least and greatest elements, and so σ will be valid in the r.e. tt-degrees. Next suppose that σ is valid in the r.e. tt-degrees and that $\mathbf{a}_1, \ldots, \mathbf{a}_n$ is a sequence satisfying φ. We construct a partial order $\mathscr{P}$ containing $\mathbf{a}_1, \ldots, \mathbf{a}_n, \mathbf{0}, \mathbf{1}$ and enough other r.e. tt-degrees so that if $\mathbf{b}, \mathbf{c}, \mathbf{d} \in \{\mathbf{a}_1, \ldots, \mathbf{a}_n, \mathbf{0}, \mathbf{1}\}$ and $\mathbf{b}, \mathbf{c} \leqslant \mathbf{d}$, and $\mathbf{b} \vee \mathbf{c} \neq \mathbf{d}$, then $\mathscr{P}$ contains an $\mathbf{e}$ so that $\mathbf{b}, \mathbf{c} \leqslant \mathbf{e}$ and $\mathbf{e} < \mathbf{d}$. Likewise if $\mathbf{d} \leqslant \mathbf{b}, \mathbf{c}$ and $\mathbf{b} \wedge \mathbf{c} \neq \mathbf{d}$ then $\mathscr{P}$ contains an $\mathbf{e}$ so that $\mathbf{e} \leqslant \mathbf{b}, \mathbf{c}$ and $\mathbf{d} < \mathbf{e}$. $\mathscr{P}$ will have no more than $(n + 2) + 2(n + 2)^3$ elements, and σ will be valid in $\mathscr{P}$.

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